PHYS 102 – General Physics I Midterm Exam Solution

1. Consider the charge distribution shown in the figure where a positive point charge Q is fixed at the point $y = d$ on the y-axis, and an infinite line charge with uniform positive density λ is fixed at $y = -d$ for $-\infty < x < \infty$.

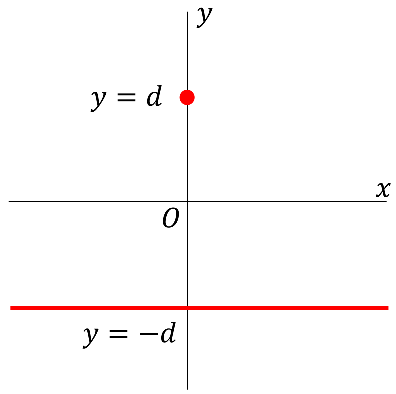
(a) (7 Pts.) What is the electric field $\vec{\boldsymbol{E}}_Q(0)$ created by the point charge at the origin?

(b) (7 Pts.) What is the electric field $\vec{\boldsymbol{E}}_\lambda(0)$ created by the line charge at the origin?

(c) (7 Pts.) What is the force $\vec{\boldsymbol{F}}_{Q\lambda}$ exerted by the point charge Q on the line charge?

(d) (7 Pts.) Find the total electric field $\vec{\boldsymbol{E}}(x)$ on the x-axis.

(e) (7 Pts.) Find the potential difference $\Delta V(x) = V(x) - V(0)$ on the x-axis.



Solution: (a)

$$\left|\vec{\boldsymbol{E}}_Q(0)\right| = \frac{1}{4\pi\epsilon_0}\frac{Q}{d^2}\,, \qquad \vec{\boldsymbol{E}}_Q(0) = \frac{1}{4\pi\epsilon_0}\frac{Q}{d^2}\,(-\hat{\mathbf{j}}).$$

(b) Use either Gauss's law with a cylindrical Gaussian surface concentric with the line charge, or integrate Coulomb's law to get

$$\vec{\boldsymbol{E}}_\lambda(0) = \frac{\lambda}{2\pi\epsilon_0\, d}\,\hat{\mathbf{j}}.$$

(c) By Newton's third law, the of the force $\vec{\boldsymbol{F}}_{Q\lambda}$ exerted by the point charge Q on the line charge is equal in magnitude and opposite in direction to the force $\vec{\boldsymbol{F}}_{\lambda Q}$ exerted by the line charge on the point charge.

$$\left|\vec{\boldsymbol{F}}_{\lambda Q}\right| = Q\left|\vec{\boldsymbol{E}}_\lambda(y = d)\right| = \frac{Q\,\lambda}{2\pi\epsilon_0(2d)} \quad\to\quad \vec{\boldsymbol{F}}_{Q\lambda}(0) = \frac{Q\,\lambda}{4\pi\epsilon_0 d}(-\hat{\mathbf{j}}).$$

(d)

$$\left|\vec{\boldsymbol{E}}_Q(x)\right| = \frac{Q}{4\pi\epsilon_0(x^2+d^2)}\,, \qquad E_{Qx} = \left|\vec{\boldsymbol{E}}_Q(x)\right|\cos\theta \quad\to\quad E_{Qx} = \frac{Q\,x}{4\pi\epsilon_0(x^2+d^2)^{3/2}}$$

$$E_{Qy} = -\left|\vec{\boldsymbol{E}}_Q(x)\right|\sin\theta \quad\to\quad E_{Qy} = \frac{-Q\,d}{4\pi\epsilon_0(x^2+d^2)^{3/2}}$$

$$E_{\lambda x} = 0\,, \qquad E_{\lambda y} = \frac{\lambda}{2\pi\epsilon_0\, d}$$

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_Q(x) + \vec{\boldsymbol{E}}_\lambda(x) = \left[\frac{Q\,x}{4\pi\epsilon_0(x^2+d^2)^{3/2}}\right]\hat{\mathbf{i}} + \left[\frac{\lambda}{2\pi\epsilon_0\, d} - \frac{Q\,d}{4\pi\epsilon_0(x^2+d^2)^{3/2}}\right]\hat{\mathbf{j}}.$$

(e) Line charge does not contribute to the potential difference because its electric field is in the y-direction.

$$V(x) - V(0) = -\int_0^x\left(\vec{\boldsymbol{E}}(x')\cdot\hat{\mathbf{i}}\right)dx' = \frac{-Q}{4\pi\epsilon_0}\int_0^x\frac{x'dx'}{\left(x'^2+d^2\right)^{3/2}} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+d^2}} - \frac{1}{d}\right).$$

2. A solid metal sphere of radius b has an empty region (void), which is also spherical with radius a. The void does not contain the center of the sphere as shown in the cross-sectional figure. The metal has **no net charge** but a positive charge Q is placed at the center of the void.

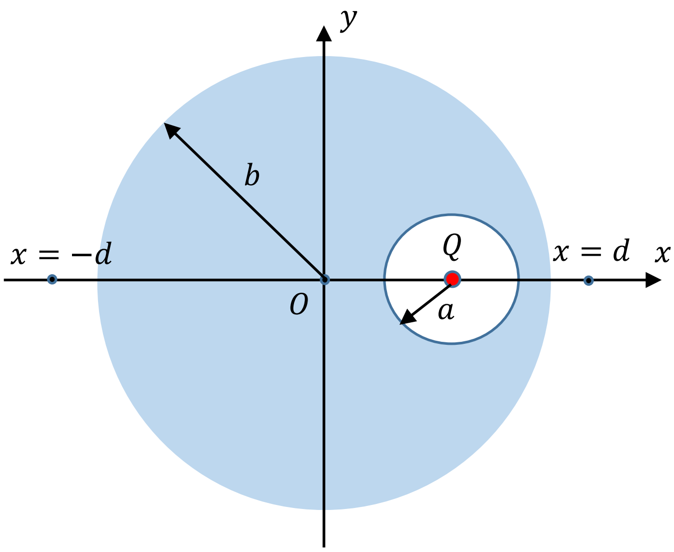
(a) (7 Pts.) What is the electric field at the center of the sphere (point O)?

(b) (7 Pts.) What is the total electric charge accumulated on the inner surface of the void?

(c) (7 Pts.) What is the charge density σ on the outer surface of the metal sphere?

(d) (7 Pts.) What is the magnitude of the electric field outside the sphere at the point $x = +d$, $(d > b)$?

(e) (7 Pts.) What is the magnitude of the electric field outside the sphere at the point $x = -d$?



Solution:

(a) Inside the conductor $\vec{\boldsymbol{E}} = 0$.

(b) Consider a spherical Gaussian surface with radius $r > a$ in the conductor, concentric with and just enclosing the void. Since $\vec{\boldsymbol{E}} = 0$ in the conductor,

$$\Phi_E = \oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = 0 \quad \rightarrow \quad Q_{\text{enc}} = 0 \quad \rightarrow \quad Q_{\text{acc}} = -Q.$$

(c) Since the metal has no net charge, charge induced on the outer surface is $+Q$, and is uniformly distributed (conducting surface).

$$\sigma = \frac{Q}{4\pi b^2}.$$

Electric field on the outer surface of the conductor is perpendicular to the surface, which is an equipotential surface. Therefore, the magnitude of the electric field on the surface is constant on the surface. If we consider a spherical Gaussian surface with radius $r > b$ outside the conductor concentric with it, we have

$$\Phi_E = \oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \rightarrow \quad E(4\pi r^2) = \frac{Q}{\epsilon_0} \quad \rightarrow \quad E = \frac{Q}{4\pi\epsilon_0 r^2}.$$

(d), (e)

$$E(-d) = E(d) = \frac{Q}{4\pi\epsilon_0 d^2}.$$

3. A capacitor is made from two parallel plates of area A, which are separated by a distance d. The region between the plates is filled by a dielectric of dielectric constant κ. The capacitor is first charged with positive charge Q, and then is disconnected, so it remains isolated throughout the problem.

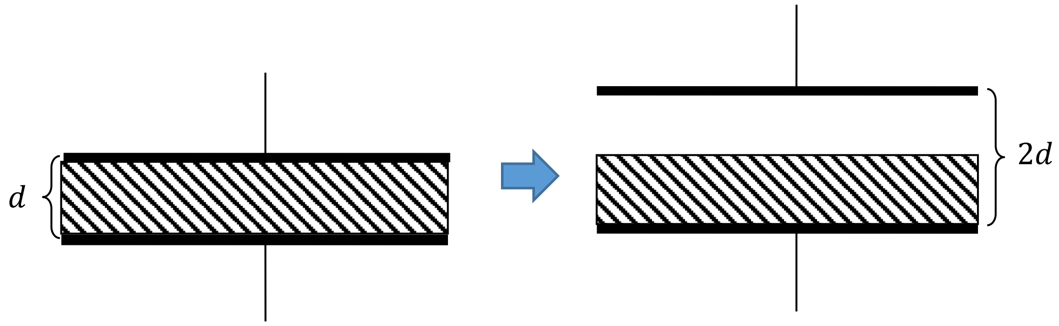
(a) (5 Pts.) What is the capacitance?

(b) (5 Pts.) What is the magnitude of the electric field between the plates?

(c) (7 Pts.) If the distance between the plates is slowly increased from d to $2d$ by applying an external force what would be the final potential difference between the plates?

(d) (9 Pts.) How much work would be done by the external force during this process?

(e) (9 Pts.) What is the magnitude of the force needed to get the top plate moving when the separation is d?



Solution: (a)

$$C = \kappa\epsilon_0 \frac{A}{d}$$

(b)

$$\Delta V = -\int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}} \quad \rightarrow \quad |\vec{\boldsymbol{E}}| d = \frac{Q}{C} \quad \rightarrow \quad E = \frac{Q}{\kappa\epsilon_0 A}.$$

(c) Distance increased but Q does not change. We can think of the final capacitor with separated plates as two capacitors connected in series.

$$\frac{1}{C_\text{f}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d}{\kappa\epsilon_0 A} + \frac{d}{\epsilon_0 A} \quad \rightarrow \quad C_\text{f} = \frac{\kappa\epsilon_0 A}{(1+\kappa)d} \quad \rightarrow \quad V_\text{f} = \frac{Q}{C_\text{f}} = (1+\kappa)\frac{Qd}{\kappa\epsilon_0 A}.$$

(d) Since the system is conservative (battery is disconnected), work done by the external force during this process is equal to the change in the electrical energy of the system.

$$U_\text{i} = \frac{Q^2}{2C_\text{i}} = \frac{Q^2 d}{2\kappa\epsilon_0 A}, \qquad U_\text{f} = \frac{Q^2}{2C_\text{f}} = \frac{Q^2(1+\kappa)d}{2\kappa\epsilon_0 A} \quad \rightarrow \quad W = \Delta U = \frac{Q^2 d}{2\epsilon_0 A}.$$

(e) Suppose we displace the top plate by a small distance Δx when the distance between the plates is d. Work done ΔW during this process can be calculated as in part (d).

$$C(d+\Delta x) = \frac{\kappa\epsilon_0 A}{d + \kappa\Delta x} \quad \rightarrow \quad U_\text{f} = \frac{Q^2(d+\kappa\Delta x)}{2\kappa\epsilon_0 A} \quad \rightarrow \quad \Delta W = \Delta U = \frac{Q^2 \Delta x}{2\epsilon_0 A}.$$

Since the displacement Δx is very small, assuming the force is constant during this interval, we have

$$\Delta W = F\,\Delta x \quad \rightarrow \quad F = \frac{\Delta W}{\Delta x} = \frac{Q^2}{2\epsilon_0 A}.$$